Nariai metric is the first example of the singularity free model

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This is just to point out that the Nariai metric is the first example of the singularity free expanding perfect fluid cosmological model satisfying the weak energy condition, $\rho > 0, \rho + p = 0$. It is a conformally non-flat Einstein space.

Following the all powerful singualrity theorems [1] and the discovery of the cosmic microwave background radiation [2] in mid sixties, the existence of singularity in the relativistic cosmological models was without question taken for granted. This view was first challenged by Senovilla's discovery [3] in 1990 of the singularity free cylindrically symmetric radiation model. The theorems did not apply in this case as the solution did not satisfy the assumption of occurrence of compact trapped surface. It is noteworthy that violation of this assumption entailed no unphysical features. This assumption seriously compromises, as is demonstrated by this example, the generality of the theorems. It has always been looked upon as suspect. From the physical point of view, this assumption could be quite acceptable for gravitational collapse but certainly not so for cosmology.

Subsquently, a non-singular family of cosmological models has been obtained [4,5]. The only two equations of state allowed are $\rho = 3p$ the radiation model [3] and $\rho = p$ the stiff fluid model [6]. Further it is also possible to include heat flux in the non-singular family without disturbing the singularity free character [7,5]. All these models were cylindrically symmetric and having diagnol metric. Mars also found a non-diagnol stiff fluid model [8].

By enlarging the scope of matter field to include imperfect fluid with anisotropic pressure and radial heat flux, a family of spherically symmetric singularity free models has been identified [9,10]. It also includes an interesting and novel case of an oscillating singularity free model [11]. The model oscillates between two regular states without ever meeting any kind of singularity anywhere. All the models satisfy the causality and the strong energy conditions with proper fall off behaviour for all the physical and kinematic parameters. The typical behaviour is to have low density at $t \to \pm\infty$ with a peak at $t = 0$. This is also the epoch of contraction turning into expansion and vice-versa.

The Nariai metric was obtained [12] in 1950 when cosmic singularity was however not much discussed about. It is an Einstein space, which implies the inflationary equation of state $\rho + p = 0$, and remarkably it is not conformally flat. It is thus not the de Sitter space. It is geodesically complete and hence is singularity free. It thus automatically satisfies the weak energy condition in the limit of the null energy condition. This is the condition which is believed to hold good always on the empirical grounds.

Let me first demonstrate the non-singular character of the metric which is given by

$$ds^2 = (1 + k^2t^2)^{-1}dt^2 - (1 + k^2t^2)dz^2 - \frac{1}{k^2}(d\Omega^2) \tag{0.1}$$

where k is a constant and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. It could be easily transformed to the form

$$ds^2 = dt^2 - \cosh^2 kt dz^2 - \frac{1}{k^2}d\omega^2 \tag{0.2}$$

by the transformation $kt \to \sinh kt$. This form clearly exhibits that spacetime is singularity free and it could be easily verified that it is geodesically complete. It has the constant density,

$$\rho = k^2 = -p \tag{0.3}$$

which could be arbitrarily chosen. It is homogeneous but anisotropic and has non-zero shear as well as non-zero Weyl curvature. It is thus not conformally flat.

The cosmic dynamics is determined by the Raychaudhuri equation [13] which reads as follows:

$$\frac{d\theta}{ds} = -4\pi(\rho + 3p) + 2\omega^2 - 2\sigma^2 - \frac{1}{3}\theta^2 + \dot{u}^a_{;a} \tag{0.4}$$

where $\theta, \sigma, \omega, \dot{u}^a$ refer respectively to expansion, shear, vorticity and acceleration. In this case, the spacetime is homogeneous and irrotational which means that the acceleration and vorticity vanish. Since the spacetime does not

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satisfy the strong energy condition, the active gravitational charge density, $\rho_c = \rho + 3p$ is in fact negative. In the above equation, shear and expansion favour contraction while $\rho_c < 0$ favours expansion. In the case of the de Sitter model, shear also vanishes and we are only left with negative ρ_c and expansion. It is geodesically incomplete as geodesics terminate at some epoch in the past. The present case essentially differs from the de Sitter by the presence of shear. It is thus the presence of shear that makes the difference and is responsible for singularity free character.

The other difference is that it is not conformally flat which is again the consequence of non-zero shear. In absence of acceleration and vorticity, the Weyl curvature is generated by shear [14]. Intutively, when shear is non-zero, the spacetime is necessarily anisotropic and consequently it cannot be conformally flat. The Weyl curvature could however be non-zero for vanishing shear spacetime. For instance the static Tolman model [15] is shear free but is conformally non-flat. The spherical model [8,10] is obtained simply by making it expand.

Note that $\theta = k \tanh kt$ and the anisotropy ratio, $(\sigma/\theta)^2 = 2/3$. It turns out that the Senovilla [3] and the spherical [9] models also have the same anisotropy ratio. The expansion parameter changes sign at $t = 0$, and it is equal to $-k$ for $t \to -\infty$ and to k for $t \to \infty$. For negative t it contracts while for positive t it expands. In contrast, it is constant for the de Sitter. This is the distinguishing feature of singularity free models. There must occur changeover from contraction to expansion and vice-versa at some finite time.

For the perfect fluid models obeying the strong energy condition, it is the acceleration (inhomogeneity) and vorticity oppose while shear (anisotropy) and expansion favour the collapse in the Raychaudhuri equation. In cosmology, vorticity is not sustainable and hence for avoidance of singualrity inhomogeneity is necessary. It though turns out that acceleration alone is never sufficient to check the collapse into singularity. There is no general result proving this but all the known cases bear it out. I would like to conjecture that it has always to be aided by shear and/or heat flux. Though shear contributes positively to collapse in the Raychaudhuri equation, it is its dynamical action which plays the crucial role of making collapse incoherent. Consequently it goes on to distracting concentration of large mass in small enough a region which is critically required for formation of trapped surface leading to singularity. Similar is the case with the heat flux [9]. It is thus necessary to have shear and/or heat flux non-zero for singularity free cosmological models. The perfect fluid singularity free models [3-8] are all both accelerating and shearing while the imperfect fluid spherical models [9-11] are all accelerating, shearing and/or also having radial heat flux.

It is clear that so long as we adhere to the strong energy condition $\rho_c > 0$, there cannot occur a singularity free homogeneous perfect fluid model. The present metric indicates that if we instead adhere only to the weak energy condition (in the null energy condition limit), we could have homogeneous but anisotropic singularity free model. Since the gravitational charge density $\rho_c < 0$, it would imply expansion. This property is however shared by the singular de Sitter and the non-singular Nariai metrics. What is then required for non-singularity is just to make the expansion incoherent so that the geodesics donot terminate in the past. This is precisely what the shear does in this case and renders the model singularity free.

The Nariai metric belongs to a spacetime which arises out of product of 2-spaces of constant curvature. In this product spacetime, when the two curvatures are equal, it is the conformally non-flat Einstein space described by the Nariai metric. When they are equal but opposite in sign, it is the conformally flat Bertotti-Robinson metric describing the uniform electric field [16,17]. In the same framework, we could have a non-flat metric with one 2-space being flat while the other 2-sphere having the constant curvature. It would then describe a cloud of string dust of uniform energy density [18]. The remarkable feature of this framework is that it gives examples of the metrics which have unusual character, opposite to what is generally the case. One associates conformal flatness with the Einstein space while conformal non-flatness with electromagnetic source. Here we have the opposite, the Einstein space is not conformally flat while the one with the uniform electric field is.

From the point of view of application to cosmology, it presents rather a queer scenario. There is inflationary expansion only in one direction while the other two remain locked to some constant value determined by the energy density. The important point to note is that there exists an inflationary solution other than the de Sitter spacetime. This is again to indicate that the de Sitter is not unique. It is indeed very important to recognise the fact that there could exist an alternative to the de Sitter inflation. Furthermore it has though been shown that it is dynamically, like the Einstein universe, not stable [19]. On perturbation, it goes over to the de Sitter spacetime.

Another remarkable feature is that it has non-zero Weyl curvature which describes free gravitational field. It would thus be physically very different from the usual de Sitter inflation. The essential difference between the two is shear. As mentioned earlier that in this case it is the shear that produces the Weyl curvature and hence it is directly related to the free gravitational field. As shear seems to be quite a generic feature of the singularity free models which suggests that free gravitational field plays significant role in avoidance of cosmic sularity. That is singularity free models must necessarily be conformally non-flat. This would be a necessary condition but however not sufficient.

Thus the difference between the Nariai and the de Sitter spacetimes is shear. It is interesting to note that on perturbation the former tends to the latter. That means perturbations tend to kill shear and thereby take the Nariai metric to the shear free de Sitter spacetime. All this exhibits an interesting interplay of shear, isotropy, Weyl curvature and occurrence of singularity.

If we wish to consider a slightly more general case than that of the perfectly homogeneous and isotropic universe, what is it that could be included without seriously disturbing the overall scenario? Obviously it would be shear and anisotropy, yet retaining homogeneity. The present case would then become pertinent as it gives inflation with shear. For this we have to go off spherical symmetry, because the de Sitter is the unique spherically symmetric Einstein space solution. The present spacetime does contain a sphere of constant radius. It could thus perhaps be considered anisotropic generalization of the de Sitter space. We would like to say that it falls within the pertinent generalization space of the standard inflationary model. It would be interesting to examine whether it could provide a tenable alternative inflationary scenario.

At present, we just wish to point out that the Nariai metric is the first example of the singularity free models satisfying the weak/null energy condition. Further, if the only weak/null energy condition is to be adhered to, then it is possible to have homogeneous singularity free model and the Nariai metric is an example of that.

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